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2005 J. Phys. A: Math. Gen. 38 333

(http://iopscience.iop.org/0305-4470/38/2/004)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 333–348 doi:10.1088/0305-4470/38/2/004

The $A_2^{(2)}$ Gaudin model and its associated Knizhnik–Zamolodchikov equation

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Received 17 July 2004, in final form 3 November 2004
Published 15 December 2004
Online at stacks.iop.org/JPhysA/38/333

Abstract

The semiclassical limit of the algebraic Bethe ansatz for the Izergin–Korepin 19-vertex model is used to solve the theory of Gaudin models associated with the twisted $A_2^{(2)}$ R-matrix. We find the spectra and eigenvectors of the $N-1$-independents Gaudin Hamiltonians. We also use the off-shell Bethe ansatz method to show how the off-shell Gaudin equation solves the associated trigonometric system of Knizhnik–Zamolodchikov equations.

PACS numbers: 05.20.−y, 05.50.+q, 04.20.Jb

1. Introduction

The exact solubility of most of classical statistical mechanics [1, 2] and two-dimensional quantum field theory models [3] relies on the $\mathcal{R}$-matrix $\mathcal{R}(u)$, where u is a spectral parameter, acting on the tensor product $V \otimes V$ of a given vector space V. $\mathcal{R}(u)$ abridges the Boltzman weights of classical statistical mechanics models or the two-body scattering matrix of integrable quantum field theories and is a solution of the Yang–Baxter (YB) equation

$$\mathcal{R}_{12}(u)\mathcal{R}_{13}(u+v)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u+v)\mathcal{R}_{12}(u), \tag{1.1}$$

in $V^1 \otimes V^2 \otimes V^3$, where $\mathcal{R}_{12} = \mathcal{R} \otimes 1, \mathcal{R}_{23} = 1 \otimes \mathcal{R}$, etc. $\mathcal{R}(u)$ may also depend on an additional parameter η in such a way that

$$\mathcal{R}(u, \eta) = 1 + \eta r(u) + \mathrm{o}(\eta^2), \tag{1.2}$$

where 1 is the identity operator in the space $V \otimes V$. In this case the solution $\mathcal{R}(u)$ of equation (1.1) is said to be semiclassical and the ‘classical r-matrix’ obeys the equation

$$[r_{12}(u), r_{13}(u+v) + r_{23}(v)] + [r_{13}(u+v), r_{23}(v)] = 0 \tag{1.3}$$

called the classical Yang–Baxter equation. Nondegenerate solutions of (1.3) in the tensor product of two copies of a simple Lie algebra g, $r_{ij}(u) \in \mathrm{g}_i \otimes \mathrm{g}_j, i, j = 1, 2, 3$, were classified by Belavin and Drinfeld [4] and its important role in the theory of classical completely integrable systems can be found, for instance, in [5].

In the skew-symmetric case $r_{ji}(-u)+r_{ij}(u) = 0$, the classical YB equation embodies the compatibility condition for the system of linear differential equations

$$\kappa \frac{\partial \Psi(z_1, \ldots, z_N)}{\partial z_i} = \sum_{j \neq i} r_{ij}(z_i - z_j)\Psi(z_1, \ldots, z_N) \tag{1.4}$$

in N complex variables $z_1, \ldots, z_N$ for a vector-valued function $\Psi(z_1, \ldots, z_N)$ with values in the tensor product space $V = V^1 \otimes \cdots \otimes V^N$. κ is a yet non-specified coupling constant.

In the rational case [4], very simple skew-symmetric solutions are known: $r(u) = \mathrm{C}_2/u$, where $\mathrm{C}_2 \in \mathrm{g} \otimes \mathrm{g}$ is a symmetric invariant tensor of a finite-dimensional Lie algebra g acting on a representation space V, and the above system of linear differential equations (1.4) is known as Knizhnik–Zamolodchikov (KZ) system of equations for the N-point conformal blocks of WZW conformal field theories on the sphere [6]. The trigonometric solutions of equation (1.1), or quantum group solutions, were classified by Jimbo [7], and in this case the system of equations (1.4) is named generalized system of KZ differential equations.

The Gaudin [8] Hamiltonians G_i are related to the classical r-matrices by

$$G_i = \sum_{j \neq i} r_{ij}(z_i - z_j) \tag{1.5}$$

and the condition for their commutativity is again the classical YB equation (1.3).

This interplay of the classical YB equation with conformal field theory and the theory of Gaudin Hamiltonians can be understood through the algebraic Bethe ansatz, here shortly described in the following way.

Babujian and Flume [9] showed that the algebraic Bethe ansatz [10] for the theory of the Gaudin models [8] can be obtained from the the algebraic Bethe ansatz of a non-homogeneous lattice vertex model, the appropriately expanded p-body Bethe wave vectors rendering, in the semiclassical limit, solutions to the generalized system of KZ equations. For instance, in the rational $su(2)$ example, the algebraic quantum inverse scattering method [10] allows one to write the following equation:

$$\tau(u|z)\Phi(u_{1,\ldots,}u_p) = \Lambda(u, u_1, \ldots, u_p|z)\Phi(u_1, \ldots, u_p) - \sum_{\alpha=1}^{p} \frac{\mathcal{F}_\alpha \Phi^\alpha}{u - u_\alpha}. \tag{1.6}$$

Here $\tau(u|z)$ denotes the transfer matrix of the rational vertex model in an inhomogeneous lattice acting on an N-fold tensor product of $su(2)$ representation spaces. Φ^α meaning $\Phi^\alpha = \Phi(u_1, \ldots, u_{\alpha-1}, u, u_{\alpha+1}, \ldots, u_p)$; $\mathcal{F}_\alpha(u_1, \ldots, u_p|z)$ and $\Lambda(u, u_1, \ldots, u_p|z)$ being c number functions. The vanishing of the so-called unwanted terms, $\mathcal{F}_\alpha = 0$, is enforced in the usual procedure of the algebraic Bethe ansatz by finding the parameters $u_1, \ldots, u_p$. In this case the wave vector $\Phi(u_1, \ldots, u_p)$ becomes an eigenvector of the transfer matrix with eigenvalue $\Lambda(u, u_1, \ldots, u_p|z)$. If we keep all unwanted terms, i.e. $\mathcal{F}_\alpha \neq 0$, then the wave vector Φ in general satisfies equation (1.6), named in [11] as off-shell Bethe ansatz equation (OSBAE).

There is a neat relationship between the wave vector satisfying the OSBAE (1.6) and the vector-valued solutions of the KZ equation (1.4): the general vector valued solution of the KZ equation for an arbitrary simple Lie algebra was found by Schechtman and Varchenko [12]. It can be represented as a multiple contour integral

$$\Psi(z_1, \ldots, z_N) = \oint \cdots \oint \mathcal{X}(u_1, \ldots, u_p|z)\phi(u_1, \ldots, u_p|z)\, \mathrm{d}u_1, \ldots, \mathrm{d}u_p, \tag{1.7}$$

where the complex variables $z_1, \ldots, z_N$ of (1.7) are related to the disorder parameters of the OSBAE. The vector-valued function $\phi(u_1, \ldots, u_p|z)$ is the semiclassical limit of the wave vector $\Phi(u_1, \ldots, u_p|z)$. In fact, it is the Bethe wave vector for Gaudin magnets [8], but 'off-shell'. The Bethe ansatz for the Gaudin model was derived for any simple Lie algebra by Reshetikhin and Varchenko [13]. The scalar function $\mathcal{X}(u_1, \ldots, u_p|z)$ is constructed from the semiclassical limit of the $\Lambda(u = z_k; u_1, \ldots, u_p|z)$ and the $\mathcal{F}_\alpha(u_1, \ldots, u_p|z)$ functions.

In this paper we investigate the semiclassical limit of Izegin–Korepin (IK) model [14], which corresponds to the twisted affine Lie algebra $A_2^{(2)}$ solution of [7].

The paper is organized as follows. In section 2 we present the algebraic Bethe ansatz for the $A_2^{(2)}$ vertex model. Here the inhomogeneous Bethe ansatz is read from the homogeneous case previously derived for the 19-vertex models [15, 16]. We also derive the off-shell Bethe ansatz equation for this vertex model. In section 3, taking into account the semiclassical limit of the results presented in section 2, we describe the algebraic structure of the corresponding Gaudin model. In section 4, data of the off-shell Gaudin equation are used to construct solutions of the trigonometric KZ equation. Conclusions are reserved for section 5.

2. Inhomogeneous algebraic Bethe ansatz

Besides $\mathcal{R}$, it is usual to consider the matrix $R = \mathcal{P}\mathcal{R}$ that satisfies the equation

$$R_{12}(u)R_{23}(u+v)R_{12}(v) = R_{23}(v)R_{12}(u+v)R_{23}(u). \tag{2.1}$$

We choose the regular solution of the YB equation for the IK 19-vertex model [14] to be normalized such that

$$R(u,\eta) = \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_5 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_7 & 0 & y_6 & 0 & x_3 & 0 & 0 \\ 0 & x_2 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_6 & 0 & x_4 & 0 & x_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_5 & 0 & x_2 & 0 \\ 0 & 0 & x_3 & 0 & x_6 & 0 & x_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2 & 0 & x_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 \end{pmatrix}, \tag{2.2}$$

where

$$\begin{aligned} &x_1(u) = \sinh(u+2\eta)\cosh(u+3\eta), && x_2(u) = \sinh u \cosh(u+3\eta), \\ &x_3(u) = \sinh u \cosh(u+\eta), && x_4(u) = \sinh u\cosh(u+3\eta) + \sinh 2\eta \cosh 3\eta, \\ &x_5(u) = \mathrm{e}^{u}\sinh 2\eta\cosh(u+3\eta), && y_5(u) = \mathrm{e}^{-u}\sinh 2\eta \cosh(u+3\eta), \\ &x_6(u) = \mathrm{e}^{u+2\eta}\sinh 2\eta \sinh u, && y_6(u) = -\mathrm{e}^{-u-2\eta}\sinh 2\eta\sinh u, \\ &x_7(u) = \mathrm{e}^{2u}x_1(u) - \mathrm{e}^{2u+4\eta}x_3(u), && y_7(u) = \mathrm{e}^{-2u}x_1(u) - \mathrm{e}^{-2u-4\eta}x_3(u). \end{aligned} \tag{2.3}$$

An inhomogeneous vertex model has two parameters: a global spectral parameter u and a disorder parameter z, so that the vertex weight matrix $\mathcal{R}$ depends on the difference $u - z$ and consequently the monodromy matrix defined below will be a function of the disorder parameters z_i.

The quantum inverse scattering method is characterized by the monodromy matrix $T(u|z)$ satisfying the equation

$$R(u-v)[T(u|z)\otimes T(v|z)] = [T(v|z)\otimes T(u|z)]R(u-v), \tag{2.4}$$

whose existence is guaranteed by the YB equation (1.1). $T(u|z)$ is a matrix in the space V (usually called auxiliary space) whose matrix elements are operators on the states of the quantum system (which will also be the space V in this work). The monodromy operator $T(u|z)$ is defined as an ordered product of local operators $\mathcal{L}_n$ (Lax operator), on all sites of the lattice:

$$T(u|z) = \mathcal{L}_N(u - z_N)\mathcal{L}_{N-1}(u - z_{N-1}) \dots \mathcal{L}_1(u - z_1). \tag{2.5}$$

We normalize the Lax operator on the nth quantum space to be given by

$$\mathcal{L}_n = \frac{1}{x_2}\begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & x_6 & 0 & x_7 & 0 & 0 \\ 0 & y_5 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_6 & 0 & x_4 & 0 & x_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2 & 0 & x_5 & 0 \\ 0 & 0 & x_7 & 0 & y_6 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_5 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 \end{pmatrix}$$

$$= \begin{pmatrix} L_{11}^{(n)}(u - z_n) & L_{12}^{(n)}(u - z_n) & L_{13}^{(n)}(u - z_n) \\ L_{21}^{(n)}(u - z_n) & L_{22}^{(n)}(u - z_n) & L_{23}^{(n)}(u - z_n) \\ L_{31}^{(n)}(u - z_n) & L_{32}^{(n)}(u - z_n) & L_{33}^{(n)}(u - z_n) \end{pmatrix}. \tag{2.6}$$

$L_{\alpha\beta}^{(n)}(u)$, $\alpha, \beta = 1, 2, 3$ are 3 by 3 matrices acting on the nth site of the lattice, so the monodromy matrix has the form

$$T(u|z) = \begin{pmatrix} A_1(u|z) & B_1(u|z) & B_2(u|z) \\ C_1(u|z) & A_2(u|z) & B_3(u|z) \\ C_2(u|z) & C_3(u|z) & A_3(u|z) \end{pmatrix}, \tag{2.7}$$

where

$$T_{ij}(u|z) = \sum_{k_1,\dots,k_{N-1}=1}^{3} L_{ik_1}^{(N)}(u - z_N) \otimes L_{k_1k_2}^{(N-1)}(u - z_{N-1}) \otimes \cdots \otimes L_{k_{N-1}j}^{(1)}(u - z_1).$$
$$i, j = 1, 2, 3. \tag{2.8}$$

The vector in the quantum space of the monodromy matrix $T(u|z)$ that is annihilated by the operators $T_{ij}(u|z), i > j$ ($C_i(u|z)$ operators, $i = 1, 2, 3$) and it is also an eigenvector for the operators $T_{ii}(u|z)$ ($A_i(u|z)$ operators, $i = 1, 2, 3$) is called a highest weight vector of the monodromy matrix $T(u|z)$.

The transfer matrix $\tau(u|z)$ of the corresponding integrable spin model is given by the trace of the monodromy matrix in the space V

$$\tau(u|z) = A_1(u|z) + A_2(u|z) + A_3(u|z). \tag{2.9}$$

The algebraic Bethe ansatz solution for the inhomogeneous IK vertex model can be obtained from the homogeneous case [15]. The proper modification is a local shift of the spectral parameter $u \to u - z_i$, and the functions needed in the following are

$$
\begin{aligned}
&z(u) = \frac{x_1(u)}{x_2(u)} = \frac{\sinh(u+2\eta)}{\sinh u}, \qquad y(u) = \frac{x_3(u)}{y_6(u)} = -\mathrm{e}^{u+2\eta}\frac{\cosh(u+\eta)}{\sinh 2\eta}, \\
&\omega(u) = \frac{x_1(u)x_3(u)}{x_4(u)x_3(u) - x_6(u)y_6(u)} = \frac{\cosh(u+\eta)}{\cosh(u-\eta)}, \\
&\mathcal{Z}(u_k - u_j) = \begin{cases} z(u_k - u_j) & \text{if} \quad k > j \\ z(u_k - u_j)\omega(u_j - u_k) & \text{if} \quad k < j. \end{cases}
\end{aligned}
\tag{2.10}
$$

We start by defining the highest weight vector of the monodromy matrix $T(u|z)$ in a lattice of N sites as the completely unoccupied state

$$
|0\rangle = \otimes_{a=1}^{N} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_a . \tag{2.11}
$$

Using (2.8) we can compute the normalized action of the monodromy matrix entries on this state

$$
\begin{aligned}
&A_i(u|z)|0\rangle = X_i(u|z)|0\rangle, \qquad C_i(u|z)|0\rangle = 0, \qquad B_i(u|z)|0\rangle \neq \{0, |0\rangle\}, \\
&X_i(u|z) = \prod_{a=1}^{N} \frac{x_i(u - z_a)}{x_2(u - z_a)}, \qquad i = 1, 2, 3.
\end{aligned}
\tag{2.12}
$$

The Bethe vectors are defined as normal ordered states $\Psi_n(u_1, \ldots, u_n)$ which can be written with the aid of the recurrence formula [15]:

$$
\begin{aligned}
\Psi_n(u_1, \ldots, u_n|z) &= B_1(u_1|z)\Psi_{n-1}(u_2, \ldots, u_n|z) \\
&\quad - B_2(u_1|z)\sum_{j=2}^{n} \frac{X_1(u_j|z)}{y(u_1 - u_j)} \prod_{k=2,k\neq j}^{n} \mathcal{Z}(u_k - u_j)\Psi_{n-2}(u_2, \ldots, \hat{u}_j, \ldots, u_n|z),
\end{aligned}
\tag{2.13}
$$

with the initial condition $\Psi_0 = |0\rangle, \Psi_1(u_1|z) = B_1(u_1|z)|0\rangle$. Here $\hat{u}_j$ denotes that the parameter u_j is absent: $\Psi(\hat{u}_j|z) = \Psi(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n|z)$.

The action of the transfer matrix $\tau(u|z)$ on the Bethe vectors gives us the following off-shell Bethe ansatz equation for the $A_2^{(2)}$ vertex model:

$$
\tau(u|z)\Psi_n(u_1, \ldots, u_n|z) = \Lambda_n\Psi_n(u_1, \ldots, u_n|z) - \sum_{j=1}^{n} \mathcal{F}_j^{(n-1)}\Psi_{(n-1)}^{j} + \sum_{j=2}^{n}\sum_{l=1}^{j-1} \mathcal{F}_{lj}^{(n-2)}\Psi_{(n-2)}^{lj}.
\tag{2.14}
$$

We now briefly describe each term which appear on the right-hand side of (2.14) (for more details the reader can see [16]): in the first term the Bethe vectors (2.13) are multiplied by c-number functions $\Lambda_n = \Lambda_n(u, u_1, \ldots, u_n|z)$ given by

$$
\Lambda_n = X_1(u|z)\prod_{k=1}^{n} z(u_k - u) + X_2(u|z)\prod_{k=1}^{n} \frac{z(u - u_k)}{\omega(u - u_k)} + X_3(u|z)\prod_{k=1}^{n} \frac{x_2(u - u_k)}{x_3(u - u_k)}. \tag{2.15}
$$

The second term is a sum of new vectors

$$
\Psi_{(n-1)}^{j} = \left(\frac{x_5(u_j - u)}{x_2(u_j - u)}B_1(u|z) - \frac{1}{y(u - u_j)}B_3(u|z)\right)\Psi_{n-1}(\hat{u}_j), \tag{2.16}
$$

multiplied by c-number functions $\mathcal{F}_j^{(n-1)}$ given by

$$\mathcal{F}_j^{(n-1)} = X_1(u_j|z)\prod_{k\neq j}^{n}\mathcal{Z}(u_k-u_j) - X_2(u_j|z)\prod_{k\neq j}^{n}\mathcal{Z}(u_j-u_k). \tag{2.17}$$

Finally, the last term is a coupled sum of a third type of vector-valued functions

$$\Psi_{(n-2)}^{lj} = B_2(u|z)\Psi_{n-2}(\hat{u}_l, \hat{u}_j), \tag{2.18}$$

with c-number coefficients

$$\begin{aligned}
\mathcal{F}_{lj}^{(n-2)} = {} & G_{lj}X_1(u_l|z)X_1(u_j|z)\prod_{k=1,k\neq j,l}^{n}\mathcal{Z}(u_k-u_l)\mathcal{Z}(u_k-u_j) \\
& + Y_{lj}X_1(u_l|z)X_2(u_j|z)\prod_{k=1,k\neq j,l}^{n}\mathcal{Z}(u_k-u_l)\mathcal{Z}(u_j-u_k) \\
& + F_{lj}X_1(u_j|z)X_2(u_l|z)\prod_{k=1,k\neq j,l}^{n}\mathcal{Z}(u_l-u_k)\mathcal{Z}(u_k-u_j) \\
& + H_{lj}X_2(u_l|z)X_2(u_j|z)\prod_{k=1,k\neq j,l}^{n}\mathcal{Z}(u_j-u_k)\mathcal{Z}(u_l-u_k)
\end{aligned} \tag{2.19}$$

where G_{lj}, Y_{lj}, F_{lj} and H_{lj} are additional functions defined by

$$\begin{aligned}
G_{lj} &= \frac{x_7(u_l-u)}{x_3(u_l-u)}\frac{1}{y(u_l-u_j)} + \frac{z(u_l-u)}{\omega(u_l-u)}\frac{x_5(u_j-u)}{x_2(u_j-u)}\frac{1}{y(u-u_l)}, \\
H_{lj} &= \frac{y_7(u-u_l)}{x_3(u-u_l)}\frac{1}{y(u_l-u_j)} - \frac{y_5(u-u_l)}{x_3(u-u_l)}\frac{1}{y(u-u_j)}, \\
Y_{lj} &= \frac{1}{y(u-u_l)}\left\{z(u-u_l)\frac{y_5(u-u_j)}{x_2(u-u_j)} - \frac{y_5(u-u_l)}{x_2(u-u_l)}\frac{y_5(u_l-u_j)}{x_2(u_l-u_j)}\right\}, \\
F_{lj} &= \frac{y_5(u-u_l)}{x_2(u-u_l)}\left\{\frac{y_5(u_l-u_j)}{x_2(u_l-u_j)}\frac{1}{y(u-u_l)} + \frac{z(u-u_l)}{\omega(u-u_l)}\frac{1}{y(u-u_j)} - \frac{x_5(u-u_l)}{x_2(u-u_l)}\frac{1}{y(u_l-u_j)}\right\}.
\end{aligned} \tag{2.20}$$

In the usual Bethe ansatz method, the next step is to be impose the vanishing of the so-called unwanted terms of (2.14) in order to get an eigenvalue problem for the transfer matrix.

We impose $\mathcal{F}_j^{(n-1)} = 0$ and $\mathcal{F}_{lj}^{(n-2)} = 0$ into (2.14) to recover the eigenvalue problem. This means that $\Psi_n(u_1,\ldots,u_n|z)$ is an eigenstate of $\tau(u|z)$ with eigenvalue Λ_n, provided the parameters u_j are solutions of the inhomogeneous Bethe ansatz equations

$$\prod_{a=1}^{N} z(u_j-z_a) = \prod_{k=1,k\neq j}^{n}\frac{z(u_j-u_k)}{z(u_k-u_j)}\omega(u_k-u_j), \qquad j=1,2,\ldots,n. \tag{2.21}$$

3. Structure of the $\mathbf{A_2^{(2)}}$ Gaudin model

In this section we will consider the theory of the Gaudin models and to do that we need to compute the expansions of some results presented in the previous section in a power series of η.

As we already mentioned the IK vertex model corresponds, in the Jimbo [7] quantum group classification, to the twisted affine Lie algebra $A_2^{(2)}$, which is constructed out of the second order automorphism of the Lie algebra A_2, the swapping of its simple roots, such that the fixed algebra under this automorphism, say g_0, is isomorphic to the Lie algebra A_1. We recall [17] the following base of 'twisted' $su(3)$ generators in the fundamental representation:

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad U^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad U^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$S^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.1}$$
$$W^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

H, U^+ and U^- correspond to the even sector of the automorphism and generate g_0, the remaining generators correspond to the odd sector and transform according to the 'isospin' 2 representation of g_0.

The quadratic Casimir operator is

$$C_2 = H^2 + \tfrac{1}{3}Y^2 + 2\{S^+, S^-\} + \{U^+, U^-\} + \{W^+, W^-\} \tag{3.2}$$

where $\{A, B\} = AB + BA$.

In order to expand the matrix elements of $T(u|z)$, up to an appropriate order in η, we will start by expanding the Lax operator entries defined in (2.6):

$$\begin{aligned}
L_{11}^{(n)} &= 1 + 2\eta \frac{\frac{1}{3}(2+Y_n) + H_n \cosh(2u-2z_n)}{\sinh(2u-2z_n)} + 4\eta^2\left(\frac{1}{2}H_n^2 - \frac{3}{4}\frac{H_n^2 - H_n}{\cosh(u-z_n)^2}\right) + o(\eta^3), \\
L_{22}^{(n)} &= 1 + 2\eta \frac{\frac{2}{3}(1-Y_n)}{\sinh(2u-2z_n)} - 4\eta^2\left(\frac{3}{4}\frac{H_n^2 - Y_n}{\cosh(u-z_n)^2}\right) + o(\eta^3), \\
L_{33}^{(n)} &= 1 + 2\eta \frac{\frac{1}{3}(2+Y_n) - H_n \cosh(2u-2z_n)}{\sinh(2u-2z_n)} + 4\eta^2\left(\frac{1}{2}H_n^2 - \frac{3}{4}\frac{H_n^2 + H_n}{\cosh(u-z_n)^2}\right) + o(\eta^3)
\end{aligned} \tag{3.3}$$

and for the elements out of the diagonal we have

$$\begin{aligned}
L_{12}^{(n)} &= 2\eta \frac{W_a^- + e^{2(u-z_n)}U_a^-}{\sinh(2u-2z_n)} + o(\eta^2), & L_{21}^{(n)} &= 2\eta \frac{W_a^+ + e^{-2(u-z_n)}U_a^+}{\sinh(2u-2z_n)} + o(\eta^2), \\
L_{23}^{(n)} &= -2\eta \frac{W_a^- - e^{2(u-z_n)}U_a^-}{\sinh(2u-2z_n)} + o(\eta^2), & L_{32}^{(n)} &= -2\eta \frac{W_a^+ - e^{-2(u-z_n)}U_a^+}{\sinh(2u-2z_n)} + o(\eta^2), \\
L_{13}^{(n)} &= 2\eta \frac{2S_n^-}{\sinh(2u-2z_n)} + o(\eta^2), & L_{31}^{(n)} &= 2\eta \frac{2S_n^+}{\sinh(2u-2z_n)} + o(\eta^2).
\end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (2.8) we will get the semiclassical expansion for the monodromy matrix entries. For the diagonal entries we get

$$A_i(u|z) = 1 + 2\eta A_i^{(1)}(u|z) + 4\eta^2 A_i^{(2)}(u|z) + o(\eta^3), \qquad i = 1, 2, 3. \tag{3.5}$$

where, the first order terms are given by

$$\begin{aligned}
A_1^{(1)}(u|z) &= \sum_{a=1}^{N} \frac{\frac{1}{3}(2+Y_a) + H_a \cosh(2u-2z_a)}{\sinh(2u-2z_a)}, \qquad A_2^{(1)}(u|z) = \sum_{a=1}^{N} \frac{\frac{2}{3}(1-Y_a)}{\sinh(2u-2z_a)}, \\
A_3^{(1)}(u|z) &= \sum_{a=1}^{N} \frac{\frac{1}{3}(2+Y_a) - H_a \cosh(2u-2z_a)}{\sinh(2u-2z_a)}
\end{aligned} \tag{3.6}$$

and the second order terms are

$$
\begin{aligned}
A_1^{(2)}(u|z) &= \sum_{a=1}^{N}\left(\frac{1}{2}H_a^2-\frac{3}{4}\frac{H_a^2-H_a}{\cosh(2u-2z_a)^2}\right)+\sum_{a<b}\coth(2u-2z_a)\coth(2u-2z_b)H_a\otimes H_b \\
&\quad+\sum_{a<b}\frac{\frac{1}{3}(2+Y_a)\otimes\frac{1}{3}(2+Y_b)}{\sinh(2u-2z_a)\sinh(2u-2z_b)}+\sum_{a<b}\frac{\frac{1}{3}(2+Y_a)\otimes H_b\cosh(2u-2z_b)}{\sinh(2u-2z_a)\sinh(2u-2z_b)} \\
&\quad+\sum_{a<b}\frac{\cosh(2u-2z_a)H_a\otimes\frac{1}{3}(2+Y_b)}{\sinh(2u-2z_a)\sinh(2u-2z_b)}+\sum_{a<b}4\frac{S_a^-\otimes S_b^+}{\sinh(2u-2z_a)\sinh(2u-2z_b)} \\
&\quad+\sum_{a<b}\frac{W_a^-+\mathrm{e}^{2(u-z_a)}U_a^-}{\sinh(2u-2z_a)}\otimes\frac{W_b^++\mathrm{e}^{-2(u-z_b)}U_b^+}{\sinh(2u-2z_b)}, \\
A_3^{(2)}(u|z) &= \sum_{a=1}^{N}\left(\frac{1}{2}H_a^2-\frac{3}{4}\frac{H_a^2+H_a}{\cosh(2u-2z_a)^2}\right)+\sum_{a<b}\coth(2u-2z_a)\coth(2u-2z_b)H_a\otimes H_b \\
&\quad+\sum_{a<b}\frac{\frac{1}{3}(2+Y_a)\otimes\frac{1}{3}(2+Y_b)}{\sinh(2u-2z_a)\sinh(2u-2z_b)}-\sum_{a<b}\frac{\frac{1}{3}(2+Y_a)\otimes H_b\cosh(2u-2z_b)}{\sinh(2u-2z_a)\sinh(2u-2z_b)} \\
&\quad-\sum_{a<b}\frac{\cosh(2u-2z_a)H_a\otimes\frac{1}{3}(2+Y_b)}{\sinh(2u-2z_a)\sinh(2u-2z_b)}+\sum_{a<b}4\frac{S_a^+\otimes S_b^-}{\sinh(2u-2z_a)\sinh(2u-2z_b)} \\
&\quad+\sum_{a<b}\left(\frac{W_a^+-\mathrm{e}^{-2(u-z_a)}U_a^+}{\sinh(2u-2z_a)}\right)\otimes\left(\frac{W_b^--\mathrm{e}^{2(u-z_b)}U_b^-}{\sinh(2u-2z_b)}\right), \\
A_2^{(2)}(u|z) &= \sum_{a=1}^{N}\left(\frac{3}{4}\frac{Y_a-H_a^2}{\cosh(2u-2z_a)^2}\right)+\sum_{a<b}\frac{\frac{2}{3}(1-Y_a)\otimes\frac{2}{3}(1-Y_b)}{\sinh(2u-2z_a)\sinh(2u-2z_b)} \\
&\quad+\sum_{a<b}\frac{W_a^++\mathrm{e}^{-2(u-z_a)}U_a^+}{\sinh(2u-2z_a)}\otimes\frac{W_b^-+\mathrm{e}^{2(u-z_b)}U_b^-}{\sinh(2u-2z_b)} \\
&\quad+\sum_{a<b}\frac{W_a^--\mathrm{e}^{2(u-z_a)}U_a^-}{\sinh(2u-2z_a)}\otimes\frac{W_b^+-\mathrm{e}^{-2(u-z_b)}U_b^+}{\sinh(2u-2z_b)}.
\end{aligned}
\tag{3.7}
$$

It will be required only to expand the off-diagonal elements up to the first order in η

$$
\begin{aligned}
B_1(u|z) &= 2\eta\sum_{a=1}^{N}\frac{W_a^-+\mathrm{e}^{2(u-z_a)}U_a^-}{\sinh(2u-2z_a)}+\mathrm{o}(\eta^2), \\
B_2(u|z) &= 2\eta\sum_{a=1}^{N}\frac{2S_a^-}{\sinh(2u-2z_a)}+\mathrm{o}(\eta^2), \\
B_3(u|z) &= -2\eta\sum_{a=1}^{N}\frac{W_a^--\mathrm{e}^{2(u-z_a)}U_a^-}{\sinh(2u-2z_a)}+\mathrm{o}(\eta^2), \\
C_1(u|z) &= 2\eta\sum_{a=1}^{N}\frac{W_a^++\mathrm{e}^{-2(u-z_a)}U_a^+}{\sinh(2u-2z_a)}+\mathrm{o}(\eta^2),
\end{aligned}
$$

$$C_2(u|z) = 2\eta \sum_{a=1}^{N} \frac{2S_a^+}{\sinh(u - z_a)} + o(\eta^2)$$

$$C_3(u|z) = -2\eta \sum_{a=1}^{N} \frac{W_a^+ - e^{-2(u-z_a)} U_a^+}{\sinh(2u - 2z_a)} + o(\eta^2). \tag{3.8}$$

From these expansions we have the following expansion for the transfer matrix (2.9):

$$\begin{aligned}\tau(u|z) = 3 + 2\eta \sum_{a=1}^{N} \frac{2}{\sinh(2u - 2z_a)} + 4\eta^2 \Bigg\{ \sum_{a=1}^{N} \left(H_a^2 - \frac{3}{2} \frac{1}{\cosh(u - z_a)^2} \right) \\ + \sum_{a<b}^{N} \frac{2}{\sinh(2u - 2z_a)\sinh(2u - 2z_b)} \Bigg\{ \frac{2}{3} + H_a \overset{s}{\otimes} H_b \cosh(2u - 2z_a) \\ \times \cosh(2u - 2z_b) + \frac{1}{3} Y_a \otimes Y_b + 2\left(S_a^+ \otimes S_b^- + S_a^- \otimes S_b^+\right) + W_a^+ \otimes W_b^- \\ + W_a^- \otimes W_b^+ + e^{2z_a - 2z_b} U_a^+ \otimes U_b^- + e^{-2z_a + 2z_b} U_a^- \otimes U_b^+ \Bigg\} \Bigg\} \\ \equiv 3 + 2\eta \tau^{(1)}(u|z) + 4\eta^2 \tau^{(2)}(u|z) + o(\eta^2).\end{aligned} \tag{3.9}$$

The second order contribution for $\tau(u|z)$ is

$$\tau^{(2)}(u|z) = \sum_{a<b}^{N} \mathcal{G}_{ab}(u) + \sum_{a=1}^{N} \left(H_a^2 - \frac{3}{2} \frac{1}{\cosh(u - z_a)^2} \right) \tag{3.10}$$

which, with the aid of the identity

$$\frac{1}{\sinh(2u - 2z_a)\sinh(2u - 2z_b)} = \frac{1}{\sinh(2z_a - 2z_b)} \left(\frac{e^{-2(u-z_a)}}{\sinh(2u - 2z_a)} - \frac{e^{-2(u-2z_b)}}{\sinh(2u - 2z_b)} \right) \tag{3.11}$$

can be written in the form

$$\tau^{(2)}(u|z) = \sum_{a=1}^{N} \frac{2\mathcal{G}_a(u)}{e^{2(u-z_a)} \sinh(2u - 2z_a)} + \sum_{a=1}^{N} \left(H_a^2 - \frac{3}{2} \frac{1}{\cosh(u - z_a)^2} \right) \tag{3.12}$$

where

$$\begin{aligned}\mathcal{G}_a(u) = \sum_{b \neq a} \frac{1}{\sinh(2z_a - 2z_b)} \Bigg\{ \frac{2}{3} + \cosh(2u - 2z_a)\cosh(2u - 2z_b) H_a \otimes H_b \\ + \frac{1}{3} Y_a \otimes Y_b + 2\left(S_a^+ \otimes S_b^- + S_a^- \otimes S_b^+\right) + W_a^- \otimes W_b^+ + W_a^+ \otimes W_b^- \\ \times e^{2(z_a - z_b)} U_a^+ \otimes U_b^- + e^{-2(z_a - z_b)} U_a^- \otimes U_b^+ \Bigg\}.\end{aligned} \tag{3.13}$$

Here we observe that $\mathcal{G}_a(u)$ is nothing but the sum of semiclassical trigonometric r-matrices. This fact follows from the construction of the semiclassical r-matrices out of the quadratic

Casimir:

$$C_2 = H^2 + \tfrac{1}{3}Y^2 + 2(S^-S^+ + S^+S^-) + (U^-U^+ + U^+U^-) + (W^+W^- + W^-W^+). \tag{3.14}$$

The Gaudin Hamiltonians are defined as the residue of $\tau(u|z)$ at the point $u = z_a$. This results in N non-local Hamiltonians

$$\begin{aligned} G_a = \sum_{b\neq a}^{N} \frac{1}{\sinh(2z_a - 2z_b)} \Bigg\{ &\frac{2}{3} + \cosh(2z_a - 2z_b) H_a \otimes H_b \\ &+ \frac{1}{3} Y_a \otimes Y_b + 2\left(S_a^+ \otimes S_b^- + S_a^- \otimes S_b^+\right) + W_a^- \otimes W_b^+ + W_a^+ \otimes W_b^- \\ &+ \mathrm{e}^{2(z_a - z_b)} U_a^+ \otimes U_b^- + \mathrm{e}^{-2(z_a-z_b)} U_a^- \otimes U_b^+ \Bigg\}, \qquad a = 1, 2, \ldots, N \end{aligned} \tag{3.15}$$

satisfying

$$\sum_{a=1}^{N} G_a = 0, \qquad \frac{\partial G_a}{\partial z_b} = \frac{\partial G_b}{\partial z_a}, \qquad [G_a, G_b] = 0, \qquad \forall a, b. \tag{3.16}$$

In the following section we will use the data of the Bethe ansatz presented in the previous section in order to find the exact spectrum and eigenvectors for each of these $(N - 1)$-independents Hamiltonians.

We complete this section deriving the $A_2^{(2)}$ Gaudin algebra from the semiclassical limit of the fundamental commutation relation (2.4): the semiclassical expansions of T and R can be written in the following form:

$$T(u|z) = 1 + 2\eta[l(u|z) + \beta(u|z)] + o(\eta^2), \qquad R(u) = P[1 + 2\eta[r(u) + \beta(u)] + o(\eta^2)] \tag{3.17}$$

where

$$\beta(u|z) = \frac{2}{3} \sum_{a=1}^{N} \frac{1}{\sinh(2u - 2z_a)}. \tag{3.18}$$

Using (3.7), (3.8) one can see that the 'classical l-operator' has the form

$$l(u|z) = \begin{pmatrix} \mathcal{Y}(u|z) + \mathcal{H}(u|z) & \mathcal{W}^-(u|z) + \mathcal{U}^-(u|z) & \mathcal{S}^-(u|z) \\ \mathcal{W}^+(u|z) + U^+(u|z) & -2\mathcal{Y}(u|z) & -\mathcal{W}^-(u|z) + \mathcal{U}^-(u|z) \\ \mathcal{S}^+(u|z) & -\mathcal{W}^+(u|z) + \mathcal{U}^+(u|z) & \mathcal{Y}(u|z) - \mathcal{H}(u|z) \end{pmatrix} \tag{3.19}$$

where

$$\begin{aligned} &\mathcal{H}(u|z) = \sum_{a=1}^{N} \coth(2u - 2z_a) H_a, \qquad \mathcal{Y}(u|z) = \sum_{a=1}^{N} \frac{\frac{1}{3}Y_a}{\sinh(2u - 2z_a)} \\ &\mathcal{W}^{\mp}(u|z) = \sum_{a=1}^{N} \frac{W_a^{\mp}}{\sinh(2u - 2z_a)}, \qquad \mathcal{U}^{\mp}(u|z) = \sum_{a=1}^{N} \frac{\mathrm{e}^{\pm 2(u - z_a)}}{\sinh(2u - 2z_a)} U_a^{\mp} \\ &\mathcal{S}^{\mp}(u|z) = \sum_{a=1}^{N} \frac{2S_a^{\mp}}{\sinh(2u - 2z_a)}. \end{aligned} \tag{3.20}$$

The corresponding semiclassical r-matrix has the form

$$\begin{aligned} r(u) = \frac{1}{\sinh 2u} \Bigg\{ &\frac{1}{3} Y \otimes Y + \cosh 2u\, H \otimes H + 2(S^+ \otimes S^- + S^- \otimes S^+) \\ &+ \mathrm{e}^{-2u} U^- \otimes U^+ + \mathrm{e}^{2u} U^+ \otimes U^- + W^- \otimes W^+ + W^- \otimes W^+ \Bigg\}. \end{aligned} \tag{3.21}$$

Here we note that (3.21) is equivalent to the r-matrix constructed out of the quadratic Casimir (3.14) in a standard way [18].

Substituting (3.21) and (3.19) into (2.4), we have

$$\begin{aligned}\mathcal{P}l(u|z)\otimes l(v|z)+\mathcal{P}r(u-v)[l(u|z)\otimes 1+1\otimes l(v|z)]\\ = l(v|z)\otimes l(u|z)\mathcal{P}+[l(v|z)\otimes 1+1\otimes l(u|z)]\mathcal{P}r(u-v),\end{aligned}\tag{3.22}$$

whose consistence is guaranteed by the classical YB equation (1.3).

From (3.22) we can derive the commutation relations between the matrix elements of $l(u|z)$. This gives us the defining relations of the $A_2^{(2)}$ Gaudin algebra:

$$\begin{aligned}
&[\mathcal{H}(u|z),\mathcal{H}(v|z)]=0, \qquad [\mathcal{H}(u|z),\mathcal{Y}(v|z)]=0,\\
&[\mathcal{H}(u|z),\mathcal{S}^{\mp}(v|z)]=\pm 2\frac{\mathcal{S}^{\mp}(u|z)-\cosh(2u-2v)\mathcal{S}^{\mp}(v|z)}{\sinh(2u-2v)},\\
&[\mathcal{H}(u|z),\mathcal{W}^{\mp}(v|z)]=\pm\frac{\mathcal{W}^{\mp}(u|z)-\cosh(2u-2v)\mathcal{W}^{\mp}(v|z)}{\sinh(2u-2v)},\\
&[\mathcal{H}(u|z),\mathcal{U}^{\mp}(v|z),]=\pm\frac{e^{\mp 2(u-v)}\mathcal{U}^{\mp}(u|z)-\cosh(2u-2v)\mathcal{U}^{\mp}(v|z)}{\sinh(2u-2v)},\\
&[\mathcal{Y}(u|z),\mathcal{Y}(v|z)]=0, \qquad [\mathcal{Y}(u|z),\mathcal{S}^{\pm}(v|z)]=0,\\
&[\mathcal{Y}(u|z),\mathcal{W}^{\pm}(v|z)]=\mp\frac{\mathcal{U}^{\pm}(u|z)-\mathcal{U}^{\pm}(v|z)}{\sinh(2u-2v)},\\
&[\mathcal{Y}(u|z),\mathcal{U}^{\pm}(v|z)]=\mp\frac{e^{\pm 2(u-v)}\mathcal{W}^{\pm}(u|z)-\mathcal{W}^{\pm}(v|z)}{\sinh(2u-2v)}\\
&[\mathcal{S}^{\pm}(u|z),\mathcal{S}^{\pm}(v|z)]=0, \qquad [\mathcal{S}^{\pm}(u|z),\mathcal{W}^{\pm}(v|z)]=0, \qquad [\mathcal{S}^{\pm}(u|z),\mathcal{U}^{\pm}(v|z)]=0\\
&[\mathcal{S}^{\pm}(u|z),\mathcal{S}^{\mp}(v|z)]=\mp 4\frac{\mathcal{H}(u|z)-\mathcal{H}(v|z)}{\sinh(2u-2v)},\\
&[\mathcal{S}^{\pm}(u|z),\mathcal{U}^{\mp}(v|z)]=\mp 2\frac{e^{\mp 2(u-v)}\mathcal{W}^{\pm}(u|z)-\mathcal{W}^{\pm}(v|z)}{\sinh(2u-2v)},\\
&[\mathcal{S}^{\pm}(u|z),\mathcal{W}^{\mp}(v|z)]=\pm 2\frac{\mathcal{U}^{\pm}(u|z)-\mathcal{U}^{\pm}(v|z)}{\sinh(2u-2v)},\\
&[\mathcal{U}^{\pm}(u|z),\mathcal{U}^{\pm}(v|z)]=0, \qquad [\mathcal{W}^{\pm}(u|z),\mathcal{W}^{\pm}(v|z)]=0,\\
&[\mathcal{U}^{\pm}(u|z),\mathcal{U}^{\mp}(v|z)]=\mp\frac{e^{\mp 2(u-v)}(\mathcal{H}(u|z)-\mathcal{H}(v|z))}{\sinh(2u-2v)},\\
&[\mathcal{U}^{\pm}(u|z),\mathcal{W}^{\pm}(v|z)]=\pm\frac{\mathcal{S}^{\pm}(u|z)-e^{\mp 2(u-v)}\mathcal{S}^{\pm}(v|z)}{\sinh(2u-2v)},\\
&[\mathcal{U}^{\pm}(u|z),\mathcal{W}^{\mp}(v|z)]=\mp 3\frac{\mathcal{Y}(u|z)-e^{\mp 2(u-v)}\mathcal{Y}(v|z)}{\sinh(2u-2v)},\\
&[\mathcal{W}^{\pm}(u|z),\mathcal{W}^{\mp}(v|z)]=\mp\frac{\mathcal{H}(u|z)-\mathcal{H}(v|z)}{\sinh(2u-2v)}.
\end{aligned}\tag{3.23}$$

A direct consequence of these relations is the commutativity of $\tau^{(2)}(u|z)$

$$[\tau^{(2)}(u|z),\tau^{(2)}(v|z)]=0, \qquad \forall u, v \tag{3.24}$$

from which the commutativity of the Gaudin Hamiltonians G_a follows immediately.

4. Off-shell Gaudin equation

In order to get the semiclassical limit of the OSBAE (2.14) we first consider the semiclassical expansions of the Bethe vectors defined in (2.13), (2.16) and (2.18):

$$
\begin{aligned}
&\Psi_n(u_1,\ldots,u_n|z) = (2\eta)^n\Phi_n(u_1,\ldots,u_n|z) + \mathrm{o}(\eta^{n+1}),\\
&\Psi^j_{(n-1)} = -2(2\eta)^{n+1}\frac{[\mathcal{U}^-(u|z)\,\mathrm{e}^{-2(u-u_j)} + \mathcal{W}^-(u|z)]}{\sinh(2u-2u_j)}\Phi_{n-1}(\hat{u}_j|z) + \mathrm{o}(\eta^{n+2}),\\
&\Psi^{lj}_{(n-2)} = (2\eta)^{n-1}\mathcal{S}^-(u|z)\Phi_{n-2}(\hat{u}_l,\hat{u}_j|z) + \mathrm{o}(\eta^n),
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{aligned}
\Phi_n(u_1,\ldots,u_n|z) &= [\mathcal{W}^-(u_1|z) + \mathcal{U}^-(u_1|z)]\Phi_{n-1}(u_2,\ldots,u_n|z)\\
&\quad + \mathcal{S}^-(u_1|z)\sum_{j=2}^{n}\frac{\mathrm{e}^{u_j-u_1}}{\cosh(u_j-u_1)}\Phi_{n-2}(u_2,\hat{u}_j,u_n|z),
\end{aligned}
\tag{4.2}
$$

with $\Phi_0 = |0\rangle$ and $\Phi_1(u_1|z) = [\mathcal{W}^-(u_1|z) + \mathcal{U}^-(u_1|z)]\Phi_0$.

The corresponding expansions of the c-number functions presented in the OSBAE are (2.14)

$$\Lambda_n = 3 + 2\eta\Lambda_n^{(1)} + 4\eta^2\Lambda_n^{(2)} + \mathrm{o}(\eta^3), \tag{4.3}$$

$$\mathcal{F}_j^{(n-1)} = 2\eta f_j^{(n-1)} + \mathrm{o}(\eta^2), \tag{4.4}$$

$$\mathcal{F}_{lj}^{(n-2)} = 2(2\eta)^3\frac{1}{\cosh(u_l-u_j)}\left\{\frac{\mathrm{e}^{u_j-u_l}f_l^{(n-1)}}{\sinh(2u-2u_l)} + \frac{\mathrm{e}^{u_l-u_j}f_j^{(n-1)}}{\sinh(2u-2u_j)}\right\} + \mathrm{o}(\eta^4), \tag{4.5}$$

where

$$\Lambda_n^{(1)} = \sum_{a=1}^{N}\frac{2}{\sinh(2u-2z_a)} \tag{4.6}$$

$$
\begin{aligned}
\Lambda_n^{(2)} &= N + \frac{3}{2}n - \frac{3}{2}\sum_{a=1}^{N}\frac{1}{\cosh(u-z_a)^2} + \frac{1}{2}\sum_{j=1}^{n}\frac{1}{\cosh(u-u_j)^2}\\
&\quad - \sum_{a=1}^{N}\sum_{j=1}^{n}\{\coth(u-z_a)\coth(u-u_j) + \tanh(u-z_a)\tanh(u-u_j)\}\\
&\quad + \sum_{a<b}^{N}\{\coth(u-z_a)\coth(u-z_b) + \tanh(u-z_a)\tanh(u-z_b)\}\\
&\quad + \sum_{j<k}^{n}\left\{\coth(u-u_j)\coth(u-u_k) + \tanh(u-u_j)\tanh(u-u_k)\right.\\
&\quad \left. + \frac{4}{\sinh(2u-2u_j)\sinh(2u-2u_k)}\right\}
\end{aligned}
\tag{4.7}
$$

and

$$f_j^{(n-1)} = \sum_{a=1}^{N}\coth(u_j-z_a) - \sum_{k\neq j}^{n}\{2\coth(u_j-u_k) - \tanh(u_j-u_k)\}. \tag{4.8}$$

Substituting these expressions into equation (2.14) and comparing the coefficients of the terms $2(2\eta)^{n+2}$ we get the first non-trivial consequence for the semiclassical limit of the OSBAE:

$$\tau^{(2)}(u|z)\Phi_n(u_1,\ldots,u_n|z) = \Lambda_n^{(2)}\Phi_n(u_1,\ldots,u_n|z) - \sum_{j=1}^{n}\frac{2f_j^{(n-1)}\Theta_{(n-1)}^{j}}{\sinh(2u-2u_j)}. \tag{4.9}$$

Note that in this limit the contributions from $\Psi_{(n-1)}^{j}$ and $\Psi_{(n-2)}^{lj}$ are combined to give a new vector valued function

$$\Theta_{(n-1)}^{j} = [\mathcal{U}^-(u|z)\,\mathrm{e}^{-2(u-u_j)} + \mathcal{W}^-(u|z)]\Phi_{n-1}(\hat{u}_j|z) + \mathcal{S}^-(u|z)\sum_{k=1,k\neq j}^{n}\frac{\mathrm{e}^{u_k-u_j}}{\cosh(u_k-u_j)}\Phi_{n-2}(\hat{u}_j,\hat{u}_k|z). \tag{4.10}$$

Finally, we take the residue of (4.9) at the point $u = z_a$ to get the off-shell Gaudin equation:

$$G_a\Phi_n(u_1,\ldots,u_n|z) = g_a\Phi_n(u_1,\ldots,u_n|z) + \sum_{l=1}^{n}\frac{2f_l^{(n-1)}\phi_{(n-1)}^{l}}{\sinh(2u_l-2z_a)}, \qquad a = 1,2,\ldots,N \tag{4.11}$$

where g_a is the residue of $\Lambda_n^{(2)}$

$$g_a = \mathrm{res}_{u=z_a}\Lambda_n^{(2)} = \sum_{b\neq a}^{N}\coth(z_a-z_b) - \sum_{l=1}^{n}\coth(z_a-u_l), \tag{4.12}$$

and $\phi_{(n-1)}^{l}$ is the residue of $\Theta_{(n-1)}^{l}$

$$\begin{aligned}\phi_{(n-1)}^{j} &= \mathrm{res}_{u=z_a}\Theta_{(n-1)}^{j}\\ &= \frac{1}{2}(\mathcal{U}_a^-\,\mathrm{e}^{2u_j-2z_a} + \mathcal{W}_a^-)\Phi_{n-1}(\hat{u}_j|z) + \mathcal{S}_a^-\sum_{k\neq j}^{n}\frac{\mathrm{e}^{u_k-u_j}}{\cosh(u_j-u_k)}\Phi_{n-2}(\hat{u}_k,\hat{u}_j|z).\end{aligned} \tag{4.13}$$

Equation (4.11) allows us solve one of the main problems of the Gaudin models, i.e., the determination of the eigenvalues and eigenvectors of the commuting Hamiltonians G_a (3.14): g_a is the eigenvalue of G_a with eigenfunction Φ_n provided u_l are solutions of the following equations $f_j^{(n-1)} = 0$, i.e.

$$\sum_{a=1}^{N}\coth(u_j-z_a) = \sum_{k\neq j}^{n}\{2\coth(u_j-u_k) - \tanh(u_j-u_k)\}, \qquad j = 1,2,\ldots,n. \tag{4.14}$$

Moreover, as we will see in the following section, the off-shell Gaudin equation (4.11) provides solutions for the differential equations known as KZ equations.

5. Knizhnik–Zamolodchikov equation

The KZ differential equation

$$\kappa\frac{\partial\Psi(z)}{\partial z_i} = G_i(z)\Psi(z), \tag{5.1}$$

appeared first as a holonomic system of differential equations of conformal blocks in WZW models of conformal field theory. Here $\Psi(z)$ is a function with values in the tensor product

$V_1 \otimes \cdots \otimes V_N$ of representations of a simple Lie algebra, $\kappa = k + g$, where k is the central charge of the associated Kac–Moody algebra, and g is the dual Coxeter number of the simple Lie algebra.

One of the remarkable properties of the KZ system is that the coefficient functions $G_i(z)$ commute and that the form $\omega = \sum_i G_i(z)\,\mathrm{d}z_i$ is closed [13]:

$$\frac{\partial G_j}{\partial z_i} = \frac{\partial G_i}{\partial z_j}, \qquad [G_i, G_j] = 0. \tag{5.2}$$

Indeed, it was indicated in [13] that equations (5.2) are not just a flatness condition for the form ω but that the KZ connection is actually a commutative family of connections.

In this section we will identify G_i with our $A_2^{(2)}$ Gaudin Hamiltonians G_a derived in the previous section

$$\begin{aligned} G_a = \sum_{b\neq a}^{N} \frac{1}{\sinh(2z_a - 2z_b)} \Big\{ &\cosh(2z_a - 2z_b) H_a \otimes H_b + \frac{1}{3} Y_a \otimes Y_b \\ &+ 2\big(S_a^+ \otimes S_b^- + S_a^- \otimes S_b^+\big) + W_a^- \otimes W_b^+ + W_a^+ \otimes W_b^- \\ &+ \mathrm{e}^{2(z_a - z_b)} U_a^+ \otimes U_b^- + \mathrm{e}^{-2(z_a - z_b)} U_a^- \otimes U_b^+ \Big\}, \qquad a = 1, 2, \ldots, N \end{aligned} \tag{5.3}$$

and show that the corresponding differential equations (5.1) can be solved via the off-shell Bethe ansatz method.

Let us now define the vector-valued function $\Psi(z_1, \ldots, z_N)$ through multiple contour integrals of the Bethe vectors (4.2)

$$\Psi(z_1, \ldots, z_N) = \oint \cdots \oint \mathcal{X}(u|z) \Phi_n(u|z)\, \mathrm{d}u_1, \ldots, \mathrm{d}u_n, \tag{5.4}$$

where $\mathcal{X}(u|z) = \mathcal{X}(u_1, \ldots, u_n, z_1, \ldots, z_N)$ is a scalar function which in this stage is still undefined.

We assume that $\Psi(z_1, \ldots, z_N)$ is a solution of the equations

$$\kappa \frac{\partial \Psi(z_1, \ldots, z_N)}{\partial z_a} = G_a \Psi(z_1, \ldots, z_N), \qquad a = 1, 2, \ldots, N \tag{5.5}$$

where G_a are the Gaudin Hamiltonians (3.14) and κ is a constant.

Substituting (5.4) into (5.5) we have

$$\kappa \frac{\partial \Psi(z_1, \ldots, z_N)}{\partial z_a} = \oint \left\{ \kappa \frac{\partial \mathcal{X}(u|z)}{\partial z_a} \Phi_n(u|z) + \kappa \mathcal{X}(u|z) \frac{\partial \Phi_n(u|z)}{\partial z_a} \right\} \mathrm{d}u, \tag{5.6}$$

where we are using a compact notation $\oint\{\circ\}\,\mathrm{d}u = \oint \ldots \oint \{\circ\}\, du_1, \ldots, \mathrm{d}u_n$.

Using the Gaudin algebra (3.23) one can derive the following non-trivial identity:

$$\frac{\partial \Phi_n}{\partial z_a} = -\sum_{l=1}^{n} \frac{\partial}{\partial u_l} \left(\frac{2\phi_{(n-1)}^l}{\sinh(2u_l - 2z_a)} \right), \tag{5.7}$$

which allows us to write (5.6) in the form

$$\begin{aligned} \kappa \frac{\partial \Psi}{\partial z_a} = \oint \left\{ \kappa \frac{\partial \mathcal{X}(u|z)}{\partial z_a} \Phi_n(u|z) + \sum_{l=1}^{n} \kappa \frac{\partial \mathcal{X}(u|z)}{\partial u_l} \left(\frac{2\phi_{(n-1)}^l}{\sinh(2u_l - 2z_a)} \right) \right\} \mathrm{d}u \\ - \kappa \sum_{l=1}^{n} \oint \frac{\partial}{\partial u_l} \left(\mathcal{X}(u|z) \frac{2\phi_{(n-1)}^l}{\sinh(2u_l - 2z_a)} \right) \mathrm{d}u. \end{aligned} \tag{5.8}$$

It is evident that the last term of (5.8) vanishes, because the contours are closed. Moreover, if the scalar function $\mathcal{X}(u|z)$ satisfies the following differential equations:

$$\kappa \frac{\partial \mathcal{X}(u|z)}{\partial z_a} = g_a \mathcal{X}(u|z), \qquad \kappa \frac{\partial \mathcal{X}(u|z)}{\partial u_j} = f_j^{(n-1)} \mathcal{X}(u|z), \tag{5.9}$$

we are recovering the off-shell Gaudin equation (4.11) from the first term in (5.8).

Taking into account the definition of the scalar functions $f_j^{(n-1)}$ (4.8) and g_a (4.12), we can see that the consistency condition of the system (5.9) is ensured by the zero curvature conditions $\partial f_j^{(n-1)} \big/ \partial z_a = \partial g_a / \partial u_j$. Moreover, the solution of (5.9) is easily obtained:

$$\mathcal{X}(u|z) = \prod_{a \neq b}^{N} \sinh(z_a - z_b)^{1/\kappa} \prod_{j \neq k}^{n} [\cosh(u_j - u_k)^{1/\kappa} \\ \times \sinh(u_j - u_k)^{-2/\kappa}] \prod_{a=1}^{N} \prod_{j=1}^{n} \sinh(z_a - u_j)^{-1/\kappa}. \tag{5.10}$$

This function determines the monodromy of $\Psi(z_1, \ldots, z_N)$ as a solution of the trigonometric KZ equation (5.5) and these results are in agreement with the Schechtman–Varchenko construction for multiple contour integral as solutions of the KZ equation in an arbitrary simple Lie algebra [12].

6. Conclusion

In this paper the $A_2^{(2)}$ 19-vertex model was investigated generalizing previous rational vertex models results relating the Gaudin magnet models to the semiclassical off-shell algebraic Bethe ansatz of these vertex models.

Using the semiclassical limit of the transfer matrix of the vertex model we derived the trigonometric $A_2^{(2)}$ Gaudin Hamiltonians. The reduction of the off-shell Gaudin equation to an eigenvalue equation gives us the exact spectra and eigenvectors for the corresponding Gaudin magnets. Data of the off-shell Gaudin equation were used to show that a hypergeometric-type integral (5.4) solves the trigonometric KZ differential equation.

Our results corroborate the method already used with success to construct solutions of trigonometric KZ equations [19–22] and elliptic KZ–Bernard equations [23].

Acknowledgment

This work was supported in part by Fundação de Amparo ă Pesquisa do Estado de São Paulo–FAPESP–Brasil, by Conselho Nacional de Desenvol-vimento–CNPq–Brasil.

References

[1] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[2] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
[3] Abdalla E, Abdalla M C B and Rothe K 2001 *Nonperturbative Methods in Two-dimensional Quantum Field Theory* 2nd edn (Singapore: World Scientific)
[4] Belavin A A and Drinfeld V G 1982 *Funct. Anal. Appl.* **16** 159
[5] Semenov-Tyan-Shanskii M A 1983 *Funct. Anal. Appl.* **17** 17
[6] Knizhnik V G and Zamolodchikov A B 1984 *Nucl. Phys.* B **247** 83
[7] Jimbo M 1986 *Commun. Math. Phys.* **102** 537
[8] Gaudin M 1976 *J. Phys.* **37** 1087

[9] Babujian H M and Flume R 1994 *Mod. Phys. Lett.* A **9** 2029
[10] Faddeev L D and Takhtajan L A 1979 *Usp. Mat. Nauk* **34** 13
[11] Babujian H M 1993 *J. Phys. A: Math. Gen.* **26** 6981
[12] Schechtman V and Varchenko A 1991 *Invent. Math.* **106** 139
[13] Reshetikhin N and Varchenko A 1994 Quasiclassical asymptotics of solutions to the KZ equations *Preprint* hep-th/9402126
[14] Izergin A G and Korepin V E 1981 *Commun. Math. Phys.* **79** 303
[15] Tarasov V O 1988 *Teor. Math. Phys.* **76** 793
[16] Lima-Santos A 1999 *J. Phys. A: Math. Gen.* **32** 1819
[17] Cornwell J F 1989 *Group Theory in Physics: vol III. Supersymmetries and Infinite-Dimensional Algebras* (New York: Academic)
[18] Kulish P P 1985 *Lett. Math. Phys.* **10** 85
[19] Babujian H M 1994 *J. Phys. A: Math. Gen.* **27** 7753
[20] Cherednik I 1991 *Publ. Res. Inst. Math. Sci.* **27** 727
[21] Lima-Santos A and Utiel W 2001 *Nucl. Phys.* B **600** 512
[22] Kulish P P and Manojlvic N 2003 *J. Math. Phys.* **44** 676
[23] Babujian H M, Poghossian R H and Lima-Santos A 1999 *Int. J. Mod. Phys.* A **14** 615